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Pseudo-Einstein real hypersurfaces in the complex quadric

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In this article, we introduce the notion of pseudo-Einstein real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$ and give a complete classification of such hypersurfaces.

1 Introduction

A Riemannian manifold M is said to be *Einstein* if the Ricci tensor Ric is a scalar multiple of the Riemannian metric g on M, that is, $g(Ric(X), Y) = \lambda g(X, Y)$ for a smooth function λ and any vector fields X, Y tangent to M. Classically, Einstein hypersurfaces in real space forms have been studied by many differential geometers.

In complex space forms or in quaternionic space forms many differential geometers have discussed real Einstein hypersurfaces, complex Einstein hypersuraces or more generally real hypersurfaces with parallel Ricci tensor, that is $\nabla Ric = 0$, where ∇ denotes the Riemannian connection of M (see Cecil-Ryan [2], Kimura [3], [4], Romero [21], [22] and Martinez and Pérez [10]).

From such a view point Kon [9] has considered the notion of pseudo-Einstein real hypersurfaces M in complex projective space $\mathbb{C}P^m$ with Kähler structure J, which are defined in such a way that

$$Ric(X) = aX + b\eta(X)\xi,$$

where a, b are constants, $\eta(X) = g(\xi, X)$ and $\xi = -JN$ for any tangent vector field X and a unit normal vector field N defined on M. In [9] Kon has also given a complete classification of pseudo-Einstein real hypersurfaces in $\mathbb{C}P^m$ by using the work of Takagi [29] and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$. Moreover, Kon [8] has considered a new notion of the Ricci tensor $\hat{Ric}$ in the generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$.

The notion of pseudo-Einstein was generalized by Cecil–Ryan [2] to any smooth functions a and b defined on M. By using the theory of tubes, Cecil–Ryan [2] have given a complete classification of such pseudo-Einstein real hypersurfaces and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$.

On the other hand, Montiel [11] considered pseudo-Einstein real hypersurfaces in complex hyperbolic space $\mathbb{C}H^m$ and gave a complete classification of such hypersurfaces and also proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}H^m$, $m \geq 3$.

For real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ the notion of pseudo Einstein was considered by Martinez and Pérez [10]. But in [15] Pérez proved that the unique Einstein real hypersurfaces in $\mathbb{H}P^m$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{1}{2m}$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The situation mentioned above is not so simple if we consider a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. This Riemannian symmetric space has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic

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Kähler manifold which is not a hyperkähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M: That $[\xi] = \text{Span }\{\xi\}$ or $\mathcal{Q}^\perp = \text{Span }\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator, being $\xi = -JN$, $\xi_i = -J_i N$, $i = 1, 2, 3$, where N denotes a unit normal vector on M in $G_2(\mathbb{C}^{m+2})$ and $\{J_1, J_2, J_3\}$ a local basis of $\mathfrak{J}$.

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *pseudo-Einstein* if the Ricci tensor Ric of M satisfies

$$Ric(X) = aX + b\eta(X)\xi + c\sum_{i=1}^{3}\eta_i(X)\xi_i$$

for any constants a, b and c on M. In a paper due to Pérez, Suh and Watanabe [19] we have defined the notion of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the assumption that b and c are *non-vanishing constants*. In this case the meaning of pseudo-Einstein is *proper pseudo-Einstein*. So in [19] we have given a complete classification of *proper Hopf* pseudo-Einstein real hypersurfaces as follows.

Theorem A *Let M be a pseudo-Einstein Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is congruent to*

(a) *a tube of radius r, $\cot^2\sqrt{2}r = \frac{m-1}{2}$, over $G_2(\mathbb{C}^{m+1})$, where $a = 4m+8$, $b + c = -2(m+1)$, provided that $c \neq -4$.*
(b) *a tube of radius r, $\cot r = \frac{1+\sqrt{4m-3}}{2(m-1)}$, over $\mathbb{H}P^m$, $m = 2n$, where $a = 8n+6$, $b = -16n+10$, $c = -2$.*

For the real hypersurfaces of type (a) or of type (b) in Theorem A the constants b and c of pseudo-Einstein real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ never both vanish on M, that is, at least one of them is non-vanishing at any point of M. As a direct consequence of Theorem A, we have also asserted that there are no Einstein Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Now let us consider the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ which is a Kähler manifold and a kind of Hermitian symmetric space of rank 2. For real hypersurfaces M in the complex quadric Q^m we have classified the isometric Reeb flow which is defined by $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ denotes Lie derivative along the Reeb direction ξ. The Lie invariant $\mathcal{L}_\xi g = 0$ along the direction ξ is equivalent to the commuting shape operator S of M in Q^m, that is, $S\phi = \phi S$. The tensor field ϕ on M is defined by $\phi X = JX - g(JX, N)N = JX - \eta(X)N$, so that ϕX is just the tangential component of JX. The classification of isometric Reeb flow was mainly used in [26], [27] and [28]. Moreover, in order to give a complete classification of pseudo-Einstein hypersurfaces in the complex quadric Q^m we need the classification of isometric Reeb flow as follows (see [26] and [27]):

Theorem B *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

The tensor field ϕ mentioned above determines the fundamental 2-form ω on a hypersurface M by $\omega(X, Y) = g(\phi X, Y)$. In this case M is said to be a contact hypersurface in a Kähler manifold if there exists an everywhere nonzero smooth function ρ on M such that $d\eta = 2\rho\omega$. It is clear that if $d\eta = 2\rho\omega$ holds then $\eta \wedge (d\eta)^{m-1} \neq 0$, that is, every contact hypersurface in a Kähler manifold is a contact manifold.

Contact hypersurfaces in complex space forms of complex dimension $m \geq 3$ have been investigated and classified by Okumura [13] (for the complex Euclidean space $\mathbb{C}^m$ and the complex projective space $\mathbb{C}P^m$) and Vernon [30] (for the complex hyperbolic space $\mathbb{C}H^m$). Now we want to introduce the following classification of contact hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ in Suh [27] and [28], which will be used in the proof of our Main Theorem in this paper:

Theorem C *Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around a totally real space form S^m in Q^m.*

Motivated by above two Theorems B and C, let us consider the notion of pseudo-Einstein real hypersurfaces in the complex quadric Q^m. When the Ricci tensor Ric of a real hypersurface M in Q^m satisfies

$$Ric(X) = aX + b\eta(X)\xi,$$

for constants $a, b \in \mathbb{R}$ and the Reeb vector field $\xi = -JN$, then M is said to be *pseudo-Einstein*.

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)AY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_z Q^m$, $z \in Q^m$ respectively.

Recall that a nonzero tangent vector $W \in T_z Q^m$, $z \in Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic,

where $V(A) = \{X \in T_{[z]} Q^{m*} \mid AX = X\}$ and $JV(A) = \{X \in T_{[z]} Q^{m*} \mid AX = -1X\}$, $[z] \in Q^{m*}$, respectively denote the $(+1)$-eigenspace and (-1)-eigenspace for the involution $A^2 = I$ on $T_{[z]} Q^{m*}$, $[z] \in Q^{m*}$.

First, in this paper we assert that any pseudo-Einstein real hypersurfaces in the complex quadric Q^m satisfies the following property:

Main Theorem 1 *Let M be a pseudo-Einstein Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the unit normal vector field N of M is singular, that is, N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic .*

Now at each point $z \in M$ let us consider the maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}_z\}$$

of $T_z M$, $z \in M$. Thus for the case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp = \text{Span}\,[A\xi, AN]$, where the complex subbundle $\mathcal{C}$ is the orthogonal complement of the Reeb vector field ξ. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space $T_z M$, $z \in M$ if the unit normal vector field N is $\mathfrak{A}$-isotropic.

When the Reeb vector field ξ satisfies $S\xi = \alpha\xi$ for the shape operator S on a real hypersurface M in the complex quadric Q^m, M is said to be *Hopf*. Then in this paper we give a complete classification for pseudo-Einstein Hopf real hypersurfaces in the complex quadric Q^m as follows:

Main Theorem 2 *Let M be a pseudo-Einstein Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then M is locally congruent to one of the following:*

(i) *M is an open part of a tube of radius r around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with $a = 2m$, and $b = -2m$.*
(ii) *$m = 2k$, M is an open part of a tube of radius r, $r = \cot^{-1}\sqrt{\frac{k}{k-1}}$ around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k} with $a = 4k$ and $b = -4 + \frac{2}{k}$.*

Now let us consider an Einstein hypersurface in the complex quadric Q^m. Then the Ricci tensor of M becomes $Ric = \lambda g$. In case (i) in above Main Theorem 2, there do not exist any Einstein hypersurfaces in Q^m, because $b = -2m$ is non-vanishing. In this case, the unit normal N on M is $\mathfrak{A}$-principal.

Moreover, in (ii), if M is assumed to be Einstein, then the constant should be $b = 0$. This gives $4 = \frac{2}{k}$, which implies a contradiction. In this case M has an $\mathfrak{A}$-isotropic unit normal vector field N in Q^m. So we conclude a corollary as follows:

Corollary 1.1 *There do not exist any Einstein Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 3$.*

2 The complex quadric

For more details in this section we refer to [6], [20], [26], [27] and [28]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For each $z \in Q^m$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [6]). The tangent space $T_z Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_z Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point z.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

In another way, the complex projective space $\mathbb{C}P^{m+1}$ can be defined by using the Hopf fibration

$$\pi : S^{2m+3} \longrightarrow \mathbb{C}P^{m+1}, \quad z \longrightarrow [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric Q^m as follows:

$$\begin{tikzcd}
\tilde{Q} = \pi^{-1}(Q) \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] & S^{2m+3} \subset \mathbb{C}^{m+2} \arrow[d, "\pi"'] \\
Q = Q^m \arrow[r, "i"] & \mathbb{C}P^{m+1}
\end{tikzcd}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is isomorphic to the oriented real 2-plane Grassmannians $G_2^+(\mathbb{R}^{m+2})$, which is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2})$, $y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^{\perp}$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} \mid g(x, u) + g(y, v) = 0,\ g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z \mid g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differental map π_* as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)} Q$ respectively. This gives that at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]} Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [20]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}} w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]} Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]} Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]} Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned}
A^2_{\lambda\bar{z}} w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda\bar{w} \\
&= \lambda A_{\bar{z}} \lambda\bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\
&= |\lambda|^2 w = w.
\end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_zQ^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X \in V(A)$, $t = 0$, there exist many kinds of maximal 2-flats $\mathbb{R}X + \mathbb{R}Z$ for $Z \in V(A)$ orthogonal to $X \in V(A)$. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$, it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by $\mathbb{R}(X + JY) + \mathbb{R}Z$ for any $Z \in V(A)$ orthogonal to $X \in V(A)$ or $\mathbb{R}(X + JY) + \mathbb{R}JZ$ for any $JZ \in JV(A)$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3 Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1 (See [26].) *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exists a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies that

$$S\xi = \alpha\xi$$

for the smooth Reeb function $\alpha = g(S\xi, \xi)$ on M. When we consider the transform JX of the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned}$$

Putting $Z = \xi$ we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 = 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) \\ - g(Y, AN)g(X, A\xi) - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Prop. 3 in Reckziegel [20]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 = 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) - g(X, A\xi)g(JY, A\xi) \\ + g(Y, A\xi)g(JX, A\xi) - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

4 Proof of Main Theorem 1

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\ & + g(SY,Z)SX - g(SX,Z)SY\end{aligned}$$

for any $X, Y, Z \in T_z M$, $z \in M$.

From this, contracting Y and Z on M in Q^m, we have for a pseudo-Einstein real hypersurface M in Q^m

$$\begin{aligned}Ric(X) &= (2m-1)X - X - \phi^2 X - 2\phi^2 X \\ &\quad - g(AN,N)AX - X + g(AX,N)AN - g(JAN,N)JAX \\ &\quad - X + g(JAX,N)JAN + (\mathrm{Tr}S)SX - S^2X \\ &= (2m-1)X - 3\eta(X)\xi - g(AN,N)AX + g(AX,N)AN \\ &\quad - g(JAN,N)JAX + g(JAX,N)JAN + (\mathrm{Tr}S)SX - S^2X \\ &= aX + b\eta(X)\xi \end{aligned} \tag{4.1}$$

where we have used the following

$$\sum_{i=1}^{2m-1} g(Ae_i,e_i) = \mathrm{Tr}A - g(AN,N) = -g(AN,N),$$

$$\sum_{i=1}^{2m-1} g(AX,e_i)Ae_i = \sum_{i=1}^{2m} g(AX,e_i)Ae_i - g(AX,N)AN = X - g(AX,N)AN,$$

$$\sum_{i=1}^{2m-1} g(JAe_i,e_i)JAX = \sum_{i=1}^{2m} g(JAe_i.e_i)JAX - g(JAN,N)JAX,$$

$$\begin{aligned}\sum_{i=1}^{2m-1} g(JAX,e_i)JAe_i &= \sum_{i=1}^{2m} g(JAX,e_i)JAe_i - g(JAX,N)JAN \\ &= JAJAX - g(JAX,N)JAN \\ &= X - g(JAX,N)JAN.\end{aligned}$$

Now in this section we want to prove our Main Theorem 1 in the introduction. In order to do this, let us put $X = \xi$ into (4.1). Then we have

$$(a+b)\xi = 2(m-2)\xi - g(AN,N)A\xi + g(A\xi,\xi)A\xi + (h\alpha - \alpha^2)\xi,$$

where the function h denotes $h = \mathrm{trace}S$. This gives

$$2g(AN,N)A\xi = \left\{2(m-2) + \left(h\alpha - \alpha^2\right) - (a+b)\right\}\xi,$$

because we have used that

$$g(A\xi,\xi) = g(AJN,JN) = -g(JAN,JN) = -g(AN,N).$$

Then it follows that

$$g(AN,N)g(A\xi,X) = 0 \tag{4.2}$$

for any vector field X orthogonal to ξ. Then (4.2) gives that $g(AN, N) = 0$ or $A\xi = \beta\xi$. From the first case we know that the unit normal vector field N is $\mathfrak{A}$-isotropic. The latter part and the involution of the complex conjugation A on Q^m give $\xi = A^2\xi = \beta A\xi = \beta^2\xi$. This gives $\beta = \pm 1$. Now let us consider $\beta = -1$. Then $A\xi = -\xi = JN$ and $A\xi = -AJN = JAN$. This means $AN = N$, that is, the unit normal N is $\mathfrak{A}$-principal.

This gives a complete proof of our Main Theorem 1 in the introduction. By virtue of this theorem, we could divide the proof of Main Theorem 2 in two cases depending on N is $\mathfrak{A}$-isotropic or N is $\mathfrak{A}$-principal. So in section 5 we give a complete classification of pseudo-Einstein real hypersurfaces in Q^m with $\mathfrak{A}$-principal normal, and in section 6 we will complete our Main Theorem 2 for the case of $\mathfrak{A}$-isotropic unit normal.

5 Pseudo-Einstein real hypersurfaces with $\mathfrak{A}$-principal normal vector field

From the expression of the curvature tensor, contracting Y and Z on M in Q^m, we have

$$\begin{aligned} Ric(X) &= (2m-1)X - X - \phi^2 X - 2\phi^2 X \\ &\quad - g(AN, N)AX - X + g(AX, N)AN - g(JAN, N)JAX \\ &\quad - X + g(JAX, N)JAN + (\mathrm{Tr}S)SX - S^2 X \\ &= (2m-1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN \\ &\quad - g(JAN, N)JAX + g(JAX, N)JAN + (\mathrm{Tr}S)SX - S^2 X. \end{aligned} \tag{5.1}$$

Now in this section we consider only an $\mathfrak{A}$-principal normal vector field N, that is, $AN = N$, for a real hypersurface M in Q^m with the notion of pseudo-Einstein. Then (5.1) becomes

$$\begin{aligned} Ric(X) &= (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2 X \\ &= aX + b\eta(X)\xi \end{aligned} \tag{5.2}$$

where $h = \mathrm{Tr}S$ denotes the mean curvature of M in Q^m, which is defined by the trace of the shape operator S on M and we have used $A\xi = -\xi$. Then from this, by differentiating the Ricci tensor, we have

$$\begin{aligned} 0 &= -(2+b)g(\nabla_X\xi, Y)\xi - (2+b)\eta(Y)\nabla_X\xi - (\nabla_X A)Y + (Xh)SY \\ &\quad + h(\nabla_X S)Y - (\nabla_X S^2)Y \\ &= -(2+b)g(\phi SX, Y)\xi - (2+b)\eta(Y)\phi SX - (\nabla_X A)Y + (Xh)SY \\ &\quad + h(\nabla_X S)Y - (\nabla_X S^2)Y, \end{aligned} \tag{5.3}$$

where $(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y$. Here, AY belongs to T_zM, $z\in M$, from the fact that $g(AY, N) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector Y on M. Then by putting $Y = \xi$ in (5.3) and using the notion of Hopf, we know that

$$\begin{aligned} (2+b)\phi SX &= -(\nabla_X A)\xi + (Xh)S\xi + h(\nabla_X S)\xi - (\nabla_X S^2)\xi \\ &= -q(X)JA\xi - \alpha\eta(X)AN + \alpha(Xh)\xi + h(\nabla_X S)\xi - (\nabla_X S^2)\xi. \end{aligned} \tag{5.4}$$

In order to get Equation (5.4) we have used the following

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= (\bar{\nabla}_X(A\xi))^T - A\nabla_X\xi \\ &= \Big\{(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi\Big\}^T - A\phi SX \\ &= q(X)JA\xi + A\phi SX + g(SX, \xi)AN - A\phi SX \\ &= q(X)JA\xi + \alpha\eta(X)AN, \end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m. Moreover, we get

$$(\nabla_X S)\xi = \nabla_X(S\xi) - S\nabla_X\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX,$$

and

$$\left(\nabla_X S^2\right)\xi = \nabla_X\left(S^2\xi\right) - S^2\nabla_X\xi = \left(X\alpha^2\right)\xi + \alpha^2\phi SX - S^2\phi SX.$$

Then (5.4) can be written as follows:

$$\begin{aligned}(2+b)\phi SX = &-q(X)JA\xi - \alpha\eta(X)AN + \alpha(Xh)\xi + h(X\alpha)\xi + h\alpha\phi SX - hS\phi SX\\ &-(X\alpha^2)\xi - \alpha^2\phi SX + S^2\phi SX.\end{aligned}$$

From this, taking the inner product with the Reeb vector field ξ, we know that the function $h\alpha - \alpha^2$ is constant. Then it can be reduced as follows:

$$(2+b)\phi SX = -q(X)JA\xi - \alpha\eta(X)AN + h\alpha\phi SX - hS\phi SX - \alpha^2\phi SX + S^2\phi SX,$$

because $h\alpha - \alpha^2$ is constant on M. From this, if we take the tangential part, we have the following:

$$\left(2+b+\alpha^2 - h\alpha\right)\phi SX = -hS\phi SX + S^2\phi SX \tag{5.5}$$

for any tangent vector $X\in T_zM$, $z\in M$, because we have assumed that the unit vector field N is $\mathfrak{A}$-principal, that is, $AN = N$, and $JA\xi = -AJ\xi = -AN$.

On the other hand, by the formula given in Suh [27] and [28] for a Hopf real hypersurface in complex quadric Q^m with $\mathfrak{A}$-principal normal vector field N, we have

$$2S\phi SX = \alpha(\phi S + S\phi)X + 2\phi X.$$

From this, it follows that

$$\begin{aligned}2S^2\phi SX &= \alpha\left(S\phi S + S^2\phi\right)X + 2S\phi X\\ &= \alpha\left(\left\{\frac{\alpha}{2}(S\phi + \phi S)X + \phi X\right\} + S^2\phi X\right) + 2S\phi X\\ &= \frac{\alpha^2}{2}(S\phi + \phi S)X + \alpha\phi X + \alpha S^2\phi X + 2S\phi X.\end{aligned} \tag{5.6}$$

Then summing up (5.5) and (5.6), we have

$$\begin{aligned}&(2+b+\alpha^2 - h\alpha)\phi SX\\ &= -h\left\{\frac{\alpha}{2}(S\phi + \phi S)X + \phi X\right\} + \frac{\alpha^2}{4}(S\phi + \phi S)X + \frac{\alpha}{2}\phi X + \frac{\alpha}{2}S^2\phi X + S\phi X.\end{aligned} \tag{5.7}$$

Remark 5.1 In Suh [26] and [27] it was proved that a real hypersurface M is a tube around an m-dimensional hypersurface S^m in Q^m if and only if the shape operator S of M satisfies $S\phi + \phi S = k\phi$ for a non-zero constant k. Then let us check that whether a tube over S^m could satisfy (5.7) or not. Then (5.7) gives

$$\left(2+b+\alpha^2 - h\alpha\right)\phi SX = -h\left\{\frac{\alpha k}{2} + 1\right\}\phi X + \frac{\alpha^2}{4}k\phi X + \frac{\alpha}{2}\phi X + \frac{\alpha}{2}S^2\phi X + S\phi X.$$

If we consider an eigen vector such that $SX = \lambda X$, then $(S\phi + \phi S)X = k\phi X$ gives that $S\phi X = (k-\lambda)\phi X$. From this, together with (5.7) and using $\alpha k = -2$ (see [26] and [27]), the principal curvatures satisfy a quadratic equation such that

$$\alpha x^2 - 2(\alpha^2 - h\alpha + 1)x = 0.$$

Then $\lambda = 0$ or $\mu = \sqrt{2}\tan\sqrt{2}r$. Moreover, the trace h of the shape operator becomes $h = \alpha + (m-1)k$. But for a tube over a sphere S^m we know that

$$\sqrt{2}\tan\sqrt{2}r = \frac{2}{\alpha}(\alpha^2 - h\alpha + 1 + b)$$

$$
\begin{aligned}
&= 2(\alpha - h) + \frac{2}{\alpha}(1 + b) \\
&= \frac{4(m-1)}{\alpha} + \frac{2}{\alpha}(1 + b) \\
&= \frac{2(2m - 1 + b)}{\alpha} \\
&= -(2m - 1 + b)\sqrt{2}\tan\sqrt{2}r,
\end{aligned}
$$

where in the third equality we have used $\alpha - h = -(m-1)k = \frac{2(m-1)}{\alpha}$ and in the fifth equality $\alpha = -\sqrt{2}\cot\left(\sqrt{2}r\right)$ respectively. This gives that $(2m + b)\sqrt{2}\tan\sqrt{2}r = 0$. So we conclude that a pseudo-Einstein real hypersurface in Q^m which is a tube over an m-dimensional sphere S^m admits $a = 2m$ and $b = -2m$. In this case the unit normal N is $\mathfrak{A}$-principal.

If we put $SX = \lambda X$, then (5.5) gives

$$(2 + \alpha^2 - h\alpha)\lambda\phi X = -h\lambda S\phi X + \lambda S^2\phi X.$$

Moreover, (5.6) gives that

$$S\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X.$$

From this, together with the above formula, we have

$$\left(2 + b + \alpha^2 - h\alpha\right)\lambda\phi X = -h\lambda\left(\frac{\alpha\lambda + 2}{2\lambda - \alpha}\right)\phi X + \lambda\left(\frac{\alpha\lambda + 2}{2\lambda - \alpha}\right)^2\phi X. \tag{5.8}$$

Now let us put $c = 2 + b + \alpha^2 - h\alpha$. By putting $X = \xi$ in (5.2) and using $\xi = \alpha\xi$, $S^2\xi = \alpha^2\xi$ and $A\xi = -\xi$, we have

$$\alpha - \alpha^2 = a + b + 2 - 2m.$$

This means that $h\alpha - \alpha^2$ is constant and $c = -(a - 2m)$.

First we consider the case that $c \neq 0$. Naturally we can consider that $\lambda \neq 0$ and $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ are both non-vanishing. Then from (5.8) the function μ satisfies the following equation

$$\mu^2 - h\mu + h\alpha - \alpha^2 - 2 - b = 0. \tag{5.9}$$

Here in (5.9) let us change the role of λ and μ conversely. Then the function λ also satisfies the following equation

$$\lambda^2 - h\lambda + h\alpha - \alpha^2 - 2 - b = 0. \tag{5.10}$$

Combining these two equations, we have

$$(\lambda - \mu)(\lambda + \mu - h) = 0.$$

Then we know that the functions λ and μ are distinct. So it implies that $h = \lambda + \mu$. Then it follows that

$$h = \lambda + \mu = \alpha + (m-1)(\lambda + \mu) = \alpha + (m-1)h,$$

this implies that $h = -\frac{\alpha}{m-2}$.

Moreover, the trace of the shape operator becomes $h = \lambda + \mu = \lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha}$. This gives

$$\lambda^2 - h\lambda + 1 + \frac{1}{2}\alpha h = 0. \tag{5.11}$$

Then (5.10) and (5.11) give

$$1 + \frac{1}{2}\alpha h = a - 2m = -\left(2 + b - h\alpha + \alpha^2\right). \tag{5.12}$$

Then $b = \frac{1}{2}h\alpha - \alpha^2 - 3$. This gives a contradiction as follows:

In fact, (5.7) with $h = -\frac{\alpha}{m-2}$ becomes

$$\left\{2 + b + \frac{(m-1)\alpha^2}{m-2}\right\}\phi SX = \frac{\alpha^2}{2(m-2)}(\phi S + S\phi)X - h\phi X + \frac{\alpha^2}{4}(S\phi + \phi S)X + \frac{\alpha}{2}\phi X + \frac{\alpha}{2}S^2\phi X + S\phi X.$$

From this, by taking the symmetric part, we get the following

$$\left\{3 + b + \frac{(m-1)\alpha^2}{m-2}\right\} g((\phi S - S\phi)X, Y) = \frac{\alpha}{2} g\left(\left(S^2\phi - \phi S^2\right)X, Y\right).$$

Then for any principal vector X such that $SX = \lambda X$ and $S\phi X = \mu\phi X$ with distinct principal curvatures λ and μ we know the following

$$(\lambda - \mu)\left\{3 + b + \frac{(m-1)\alpha^2}{m-2} - \frac{\alpha}{2}(\lambda + \mu)\right\}\phi X = 0,$$

From this, together with the fact that $\lambda + \mu = h = -\frac{\alpha}{m-2}$, it follows that

$$3 + b + \frac{\{2(m-1)+1\}\alpha^2}{2(m-2)} = 0,$$

which gives a contradiction. Because we know that

$$\begin{aligned} 0 &= 3 + b + \frac{\{2(m-1)+1\}\alpha^2}{2(m-2)} \\ &= \frac{1}{2}h\alpha - \alpha^2 + \frac{(2(m-1)+1)\alpha^2}{2(m-2)} \\ &= -\frac{\alpha^2}{2(m-2)} - \alpha^2 + \frac{\{2(m-1)+1\}\alpha^2}{2(m-2)} \\ &= \frac{1}{m-2}\alpha^2. \end{aligned}$$

Then we have $\alpha = 0$, which implies $h = \lambda + \mu = 0$ and $\mu = \frac{1}{\lambda}$. This gives a contradiction.

Next we consider the case that $c = -(a - 2m) = 0$. Then we can consider $\lambda = 0$ and $\mu = -\frac{2}{\alpha} \neq 0$. In this case, we can easily verify that $h = -\frac{2}{\alpha}$. Then from Equation (5.7), together with $h\alpha = -2$, the expression of the shape operator becomes

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -\frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

This means equivalently the shape operator satisfies $S\phi + \phi S = k\phi$, where $k = -\frac{2}{\alpha}$. Then by Theorem C in the introduction, M is a tube of radius r around a totally geodesic and totally real m-dimensional sphere S^m in Q^m. As previously mentioned, the tube is a pseudo-Einstein real hypersurface in complex quadric Q^m with $a = 2m$ and $b = -2m$.

6 Pseudo-Einstein real hypersurfaces with $\mathfrak{A}$-isotropic normal vector field

In this section we want to prove Theorem 2 for pseudo-Einstein real hypersurfaces in Q^m with $\mathfrak{A}$-isotropic unit normal vector field.

In order to do this, from the assumption of pseudo-Einstein, first we prove an important proposition as follows:

Proposition 6.1 *Let M be a pseudo-Einstein real hypersurface in complex quadric Q^m, $m \geq 3$ with $\mathfrak{A}$-isotropic unit normal. Then the distributions $\mathcal{Q}$ and $\mathcal{Q}^{\perp} = \mathcal{C} \ominus \mathcal{Q}$ are invariant by the shape operator S of M in Q^m.*

Proof. Since M is $\mathfrak{A}$-isotropic, we know that $g(A\xi, \xi) = 0$, $g(AN, N) = 0$ and $g(A\xi, N) = 0$. In this case the Ricci tensor becomes

$$\begin{aligned} Ric(X) &= (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X \\ &= aX + b\eta(X)\xi. \end{aligned} \tag{6.1}$$

In order to prove this proposition let us introduce two important formulas for an $\mathfrak{A}$-isotropic normal vector field as follows: By putting $X = A\xi$ and $X = AN$ in (6.1), we have respectively

$$0 = (2m-a)A\xi + hSA\xi - S^2A\xi$$

and

$$0 = (2m-a)AN + hSAN - S^2AN.$$

Now let us consider another operator T defined by $T = S^2 - hS$. Then T becomes a new symmetric operator. The above two equation means that $g(T\mathcal{Q}^{\perp}, \mathcal{Q}) = 0$, where $\mathcal{Q}^{\perp} = \mathcal{C} - \mathcal{Q} = [A\xi, AN]$. This gives the fact that the distribution $\mathcal{Q}^{\perp}$ is invariant by the operator T.

On the other hand, the operator T commutes with the shape operator S, that is, $TS = ST$. Then there exist a common basis which gives a simultaneous digonalization for both two symmetric operators S and T, which implies $g(S\mathcal{Q}^{\perp}, \mathcal{Q}) = 0$ for the shape operator S on M in Q^m. This means that the distribution $\mathcal{Q}^{\perp}$ is invariant by the shape operator S. $\square$

Putting $X = \xi$ into (6.1), we have

$$a + b = h\alpha - \alpha^2 + 2m - 4.$$

By Proposition 6.1, we may put $SA\xi = \beta A\xi$ and $SAN = \gamma AN$. Then by (6.1), we have respectively

$$h\beta - \beta^2 + (2m-a) = 0, \tag{6.2}$$

and

$$h\gamma - \gamma^2 + (2m-a) = 0. \tag{6.3}$$

From these two equations, we know that $(\beta - \gamma)\{h - (\beta + \gamma)\} = 0$.

On the other hand, let us consider $X \in T_\lambda \subset \mathcal{Q}$ in (6.1) such that $SX = \lambda X$, $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$. Then respectively we have

$$\lambda^2 - h\lambda + a - (2m-1) = 0, \tag{6.4}$$

and

$$\mu^2 - h\mu + a - (2m-1) = 0. \tag{6.5}$$

Substracting (6.5) from (6.4), it follows that

$$(\lambda - \mu)\{h - (\lambda + \mu)\} = 0.$$

Then in such an $\mathfrak{A}$-isotropic case we want to make the derivative of the Ricci tensor as follows:

$$\begin{aligned} (\nabla_Y Ric)X &= \nabla_Y(Ric(X)) - Ric(\nabla_Y X) \\ &= -3(\nabla_Y \eta)(X)\xi - 3\eta(X)\nabla_Y \xi \end{aligned}$$

$$\begin{aligned}&+g(X,\nabla_Y(AN))AN+g(AX,N)\nabla_Y(AN)\\&+g((\nabla_Y(A\xi),X)A\xi+\eta(AX)\nabla_Y(A\xi)+(Yh)SX\\&+h(\nabla_Y S)X-(\nabla_Y S^2)X.\end{aligned}$$

Since $AN=BN$ for an $\mathfrak{A}$-isotropic normal vector field (see [26] and [27]), we know that

$$\nabla_Y(BN)=\nabla_Y(AN)=\{(\bar\nabla_Y A)N+A\bar\nabla_Y N\}^T=\{q(Y)JAN-ASY\}^T,$$

and

$$\begin{aligned}\nabla_Y(A\xi)&=\{(\bar\nabla_Y A)\xi+A\bar\nabla_Y\xi\}^T\\&=\{q(Y)JA\xi+A\phi SY\}^T,\end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m. As M is pseudo-Einstein, the above formula becomes

$$\begin{aligned}0=&-(3+b)g(\phi SY,X)\xi-(3+b)\eta(X)\phi SY+\{q(Y)g(JAN,X)\\&-g(ASY,X)\}AN+g(AX,N)\{q(Y)JAN-ASY\}^T+\{q(Y)g(JA\xi,X)\\&+g(A\phi SY,X)\}A\xi+\eta(AX)\{q(Y)JA\xi+A\phi SY\}^T\\&+(Yh)SX+h(\nabla_Y S)X-\left(\nabla_Y S^2\right)X.\end{aligned}\tag{6.6}$$

Since the unit normal N is $\mathfrak{A}$-isotropic, we know that

$$g(\xi,A\xi)=0,\quad g(\xi,AN)=0,\quad g(AN,N)=0,\quad g(A\xi,N)=0,$$

and

$$g(JAN,\xi)=-g(AN,N)=0.$$

By taking the inner product (6.6) with the Reeb vector field ξ, we have

$$\begin{aligned}0=&-(3+b)g(\phi SY,X)+g(AX,N)\eta(ASY)+\eta(AX)\eta(A\phi SY)\\&+(Yh)\alpha\eta(X)+hg((\nabla_Y S)X,\xi)-g\left(\left(\nabla_Y S^2\right)X,\xi\right).\end{aligned}\tag{6.7}$$

On the other hand, let us use the following calculation for a Hopf hypersurface in Q^m. Then by differentiating $S\xi=\alpha\xi$, we have

$$\begin{aligned}(\nabla_X S)\xi&=(X\alpha)\xi+\alpha\phi SX-S\phi SX,\\(\nabla_X S^2)\xi&=(X\alpha^2)\xi+\alpha^2\phi SX-S^2\phi SX.\end{aligned}$$

From this, together with putting $X=\xi$ in (6.6) and using $g(\xi,AN)=g(A\xi,\xi)=0$, we have

$$\begin{aligned}(3+b)\phi SY&=(Yh)S\xi+h(\nabla_Y S)\xi-\left(\nabla_Y S^2\right)\xi-g(ASY,\xi)AN+g(A\phi SY,\xi)A\xi\\&=(Yh)\alpha\xi+h\{(Y\alpha)\xi+\alpha\phi SY-S\phi SY\}-\left\{\left(Y\alpha^2\right)\xi+\alpha^2\phi SY-S^2\phi SY\right\}\\&\quad-g(ASY,\xi)AN+g(A\phi SY,\xi)A\xi.\end{aligned}$$

From this, taking the inner product with Reeb vector field ξ, it follows that the function $h\alpha-\alpha^2$ is constant on M. Then it can be rearranged as follows:

$$(3+b+\alpha^2-\alpha h)\phi SY=-hS\phi SY+S^2\phi SY-g(ASY,\xi)AN+g(A\phi SY,\xi)A\xi.\tag{6.8}$$

Since the unit normal N is $\mathfrak{A}$-isotropic, we know that $g(\xi,A\xi)=0$. Moreover, by Lemma 4.2 in [26], we have the following

$$2S\phi SX=\alpha(S\phi+\phi S)X+2\phi X-2g(X,AN)A\xi+2g(X,A\xi)AN.\tag{6.9}$$

On the other hand, by Proposition 6.1, the distribution $\mathcal{Q}^{\perp}$ is invariant by the shape operator S. Then (6.9) gives the following for $SAN = \gamma AN$

$$\begin{aligned}(2\gamma - \alpha)S\phi AN &= (\alpha\gamma + 2)\phi AN - 2A\xi \\ &= (\alpha\gamma + 2)\phi AN - 2\phi AN \\ &= \alpha\gamma\phi AN.\end{aligned}$$

Since $A\xi = \phi AN$, we have the following

$$SA\xi = \frac{\alpha\gamma}{2\gamma - \alpha}A\xi.$$

From (6.2) and (6.3) we have two cases that $\gamma = \beta$, or $h = \gamma + \beta$, where $\beta = \frac{\alpha\gamma}{2\gamma-\alpha}$. Then the first case $\gamma = \beta = \frac{\alpha\gamma}{2\gamma-\alpha}$ gives

$$\gamma = 0 \quad \text{or} \quad \gamma = \alpha. \tag{6.10}$$

The latter case $h = \gamma + \beta$ only occurs for $\gamma \neq \beta$. This case $h = \gamma + \beta$ can be regarded as Case III which will be discussed in detail at the final part of section 6. Now by putting $Y = \phi AN$ in (6.8) such that $SY = \gamma Y$, $S\phi Y = \beta\phi Y$, $\beta = \frac{\alpha\gamma}{2\gamma-\alpha}$ we know that

$$(3 + b + \alpha^2 - \alpha h)\gamma = -h\gamma\beta + \gamma\beta^2 + \gamma.$$

Since $\gamma \neq 0$, the equation becomes

$$2 + b + \alpha^2 - \alpha h = -h\beta + \beta^2.$$

From this, together with $h = \gamma + \beta$, it follows that

$$\alpha\gamma^2 - 2(2 + b + \alpha^2)\gamma + (2 + b + \alpha^2)\alpha = 0 \tag{6.11}$$

for $\gamma \neq 0$ and using the equation $h = \gamma + \frac{\alpha\gamma}{2\gamma-\alpha}$, that is, $\gamma^2 - h\gamma + \frac{1}{2}\alpha h = 0$. So by using this equation, in Case III we will show that the latter case $h = \gamma + \beta$ for $\gamma \neq \beta$ can not happen.

On the other hand, on the distribution $\mathcal{Q}$ we know that $AX \in T_zM$, $z \in M$, because $AN \in \mathcal{Q}^{\perp}$. So (6.9), together with the fact that $g(X, A\xi) = 0$ and $g(X, AN) = 0$ for any $X \in \mathcal{Q}$, imply that

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \tag{6.12}$$

Then we can take an orthonormal basis $X_1, \ldots, X_{2(m-2)} \in \mathcal{Q}$ such that $AX_i = \lambda_i X_i$ for $i = 1, \ldots, m-2$. Then by (6.9) we know that

$$S\phi X_i = \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\phi X_i.$$

Accordingly, by (6.10) and (6.12) the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{m-2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_{m-2} \end{bmatrix}$$

So on the distribution $\mathcal{Q}$, for any $X, \phi X \in \mathcal{Q}$ such that $SX = \lambda X$ and $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, (6.8) becomes the following

$$\begin{aligned}(3+b+\alpha^2-\alpha h)\lambda\phi X &= -h\lambda S\phi X + \lambda S^2\phi X + \lambda g(X, A\xi)AN + \lambda g(A\phi X, \xi)A\xi \\ &= -h\lambda S\phi X + \lambda\mu^2\phi X \\ &= -h\lambda\mu\phi X + \lambda\mu^2\phi X,\end{aligned} \tag{6.13}$$

where we have used $\mathcal{C} - \mathcal{Q} = \text{Span}\,[A\xi, AN]$ in the second equality. From this, if there exists a non-vanishing principal curvature $\lambda \neq 0$, then any principal curvature of the shape operator on the distribution Q satisfies the following quadratic equation

$$x^2 - hx + (\alpha h - \alpha^2 - 3 - b) = 0. \tag{6.14}$$

Summing up the above discussions including the first and the latter cases, we can divide into the following three Cases I, II and III.

Case I. $\beta = \gamma(=\alpha) = 0$

In this case we know $a = 2m$. Accordingly, by (6.4) and (6.5) the shape operator S can be expressed by

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

Now we consider the following subcase

Subcase 1.1. $\lambda = \mu$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.

In this subcase the solutions $\cot r$ and $-\tan r$ are roots of the quadratic equation $x^2 - \alpha x - 1 = 0$. Then M becomes a real hypersurface with isometric Reeb flow. Then by the results mentioned in Suh [26] and [27], m is even, say $m = 2k$ for a natural number k, and the real structure A maps the principal curvature space $T_{\cot r}$ and $T_{\tan r}$, that is, $AT_{\cot r} = T_{\tan r}$. So we should have the multiplicities of the principal curvatures $\cot r$ and $\tan r$ become $2k-2$ respectively. Moreover, the two roots $\cot r$ and $\tan r$ respectively satisfy

$$\cot^2 r - h\cot r + 1 = 0$$

and

$$\tan^2 r + h\tan r + 1 = 0.$$

From the first equation we know

$$h\cot r = 1 + \cot^2 r.$$

Moreover, the trace of the shape operator h becomes

$$\begin{aligned}h &= \alpha + (2k-2)(\cot r - \tan r) \\ &= (2k-1)\alpha \\ &= (2k-1)(\cot r - \tan r).\end{aligned}$$

From these two equations we have

$$\begin{aligned}h\cot r &= (2k-1)(\cot r - \tan r)\cot r \\ &= (2k-1)\cot^2 r - (2k-1) \\ &= 1 + \cot^2 r.\end{aligned}$$

This gives $\cot^2 r = \frac{k}{k-1}$. Moreover, from the expression of the shape operator for $\lambda = \cot r$ and $\mu = -\tan r$, the structure tensor ϕ commutes with the shape operator S of M in Q^m, that is $S\phi = \phi S$. Then by the result due to Suh [26] and [27], M is locally congruent to a tube of radius $r = \cot^{-1}\sqrt{\frac{k}{k-1}}$ around $\mathbb{C}P^k$ is a pseudo-Einstein real hypersurface in Q^{2k} with $a = 4k$ and $b = -4 + \frac{2}{k}$.

Subcase 1.2. $h = \lambda + \mu$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.

When the principal curvatures λ and μ are different from each other, then we should have $h = \lambda + \mu = \lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then naturally the root λ satisfies the quadratic equation

$$\lambda^2 - h\lambda + \frac{1}{2}\alpha h + 1 = 0.$$

From this, together with (6.4), it follows that $\alpha h = 0$, which gives $\alpha = 0$ or $h = 0$.

For the case where the function $\alpha = 0$, then $\mu = \frac{1}{\lambda}$. Then the trace $h = \alpha + (m-2)(\lambda + \mu) = \alpha + (m-2)h$ gives $h = -\frac{\alpha}{m-3} = 0$. But in this subcase the trace becomes $h = \lambda + \frac{1}{\lambda} = 0$. This gives a contradiction. Moreover, for the case $h = 0$, we also get $\lambda^2 + 1 = 0$, which gives a contradiction.

Case II. $\beta = \gamma(= \alpha)$.

In such a case the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

Then the roots λ and μ are solutions of the quadratic equation

$$x^2 - hx + a - (2m-1) = 0.$$

This gives

$$(\lambda - \mu)(\lambda + \mu - h) = 0.$$

Now let us consider two subcases as follows:

Subcase 2.1. $\lambda = \mu$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.

In this subcase we know that $\cot r$ and $-\tan r$ are solutions of the equation $x^2 - hx + a - (2m-1) = 0$. Then even in this subcase M becomes a real hypersurface with isometric Reeb flow. Then also by using the result in Suh [26] and [27], m is even, say $m = 2k$ for a natural number k, and the real structure A maps the principal curvature space as $AT_{\cot r} = T_{\tan r}$. So we should have the multiplicities of the principal curvatures $\cot r$ and $\tan r$ become $2k-2$ respectively.

So it follows that

$$\cot^2 r - h\cot r + a - (2m-1) = 0,$$

and

$$\tan^2 r + h\tan r + a - (2m-1) = 0.$$

Then the trace h becomes

$$\begin{aligned} h &= 3\alpha + (2k-2)(\cot r - \tan r) \\ &= (2k+1)(\cot r - \tan r). \end{aligned}$$

From the above equations we see that

$$\begin{aligned} h \cot r &= (2k+1)(\cot r - \tan r)\cot r \\ &= \cot^2 r + a - (2m-1) \\ &= (2k+1)\cot^2 r - (2k+1). \end{aligned}$$

This implies

$$\cot^2 r = \frac{(2k+1) + a - (2m-1)}{2k}.$$

Moreover, we know that $\gamma\beta + (2m-a) = \alpha^2 + (2m-a) = 0$. This gives $\alpha^2 = \cot^2 r + \tan^2 r - 2 = a - 2m$. From this, together with the above equation, we have

$$(2k-1)\cot^4 r - 2k\cot^2 r - 1 = 0.$$

Then it follows that

$$\cot^2 r = \frac{k + \sqrt{k^2+2k-1}}{2k-1}. \tag{6.15}$$

In this subcase the shape operator S takes the form

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix}$$

This means that the shape operator S commutes with the structure tensor ϕ. Then by Suh [26] and [27] we have the function α should be vanishing, that is, $\alpha = \cot r - \tan r = 0$. Then the radius r becomes $r = \frac{\pi}{4}$. From this, together with (6.15), we have $2k-1 = k + \sqrt{k^2+2k-1}$. This implies $k = \frac{1}{2}$, which gives a contradiction. So such a case cannot happen.

Subcase 2.2. $h = \lambda + \mu$
In this subcase we have

$$\begin{aligned} h &= 3\alpha + (m-2)(\lambda+\mu) \\ &= \lambda + \mu. \end{aligned}$$

Then $h = -\frac{3\alpha}{m-3}$. This gives $\lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha} = -\frac{3\alpha}{m-3}$, which implies

$$2(m-3)\lambda^2 + 6\alpha\lambda + \{2(m-3) - 3\alpha^2\} = 0.$$

From this, together with (6.4), we know that

$$a = 2m - \frac{3\alpha^2}{2(m-3)}. \tag{6.16}$$

On the other hand, by the assumption of $\alpha = \beta = \gamma$, we have

$$a = h\alpha - \alpha^2 + 2m.$$

From this, together with (6.16), it follows that

$$h\alpha - \alpha^2 + 2m = 2m - \frac{3\alpha^2}{2(m-3)}.$$

This gives that $(2m-3)\alpha^2 = 0$, which implies $\alpha = 0$. Then $h = 0$ gives $\lambda = -\mu = -\frac{1}{\lambda}$. Then we get a contradiction. So such a subcase does not appear.

Case III. $h = \beta + \gamma, \beta \neq \gamma$.

As mentioned previously, let us consider (6.11). In such a case the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

Then $h = \beta + \gamma = \gamma + \frac{\alpha\gamma}{2\gamma - \alpha}$ gives the following equation

$$\gamma^2 - h\gamma + \frac{1}{2}\alpha h = 0.$$

So two roots γ and β are solutions of the equation $x^2 - hx + \frac{1}{2}\alpha h = 0$ with

$$\gamma\beta = a - 2m = \frac{1}{2}\alpha h, \tag{6.17}$$

where in the first equality we have used the assumption $h = \beta + \gamma$ from (6.2). Then we could divide this into two subcases as follows:

Subcase 3.1. $h = \lambda + \mu$

Then both roots λ and μ are the solutions of the following equation

$$x^2 - hx + \frac{1}{2}\alpha h + 1 = 0.$$

On the other hand, let us write (6.14) again as follows:

$$x^2 - hx + (\alpha h - \alpha^2 - 3 - b) = 0.$$

Comparing these two equations, we know that M is a pseudo-Einstein real hypersurface satisfying

$$a = 2m + \frac{1}{2}\alpha h \quad \text{and} \quad b = \frac{1}{2}\alpha h - \alpha^2 - 4, \tag{6.18}$$

because we know that

$$a + b = h\alpha - \alpha^2 + 2m - 4.$$

On the other hand, the trace h can be written as follows:

$$\begin{aligned} h &= \lambda + \mu \\ &= \beta + \gamma \\ &= \alpha + \beta + \gamma + 2(m-2)(\lambda + \mu) \\ &= \alpha + h + 2(m-2)h \\ &= \alpha + (2m-3)h. \end{aligned}$$

Then the trace becomes

$$h = -\frac{\alpha}{2(m-2)}. \tag{6.19}$$

From this, together (6.11) and the equation $x^2 - hx + \frac{1}{2}\alpha h = 0$, we have

$$h = \frac{2(\alpha^2 + b + 2)}{\alpha} = -\frac{\alpha}{2(m-2)}.$$

Then it follows that

$$b = -\frac{\alpha^2}{4(m-2)} - \alpha^2 - 2. \tag{6.20}$$

On the other hand, in Case 3 we have already mentioned that the constant b is given in (6.18). Then (6.18) and (6.19) give

$$b = \frac{1}{2}\alpha h - \alpha^2 - 4 = -\frac{\alpha^2}{4(m-2)} - \alpha^2 - 4. \tag{6.21}$$

So, if we compare the two constants b in (6.20) and (6.21), we get a contradiction. So this case can not occur.

Subcase 3.2. $\lambda = \mu$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.
Then $\lambda = \mu = \cot r$ or $\lambda = \mu = -\tan r$, $\alpha = 2\cot 2r$. Of course, the trace h becomes the following

$$\begin{aligned} h &= \beta + \gamma \\ &= \alpha + \beta + \gamma + (2k-2)(\cot r - \tan r). \end{aligned}$$

Then this gives $(2k-1)\alpha = 0$, which implies $\alpha = 0$, that is, $r = \frac{\pi}{4}$. Moreover, by (6.17), $\alpha = 0$ gives $a = 2m$ and $\gamma\beta = 0$. So this implies $\beta = \frac{\alpha\gamma}{2\gamma-\alpha} = 0$ and $\gamma = \frac{\alpha\beta}{2\beta-\alpha} = 0$. This gives a contradiction, because we have assumed $\beta \neq \gamma$.

Summing up all Subcases 3.1, 3.2, and 3.3, we note that Case III does not appear.

Remark 6.2 In this remark let us check that whether the Ricci tensor of the tube M over a totally geodesic $\mathbb{C}P^k$ in Q^m, $m = 2k$, mentioned in Suh [26] and [27] satisfies the notion of pseudo-Einstein or not. Then by Theorem B in the introduction, the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. In this case we know that the normal vector field N of M in Q^{2k} is $\mathfrak{A}$-isotropic. Then $g(AN, N) = 0$, $g(A\xi, \xi) = 0$, $g(A\xi, N) = 0$. So let us suppose that M is pseudo-Einstein. Then for any vector field X on M the Ricci tensor Ric becomes

$$\begin{aligned} Ric(X) &= (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X \\ &= aX + b\eta(X)\xi \end{aligned} \tag{6.22}$$

for some constant real numbers a and b. Putting $X = \xi$ into (6.22), we have

$$(a+b)\xi = (2m-4)\xi + (h\alpha - \alpha^2)\xi,$$

where

$$\begin{aligned} h\alpha - \alpha^2 &= \{2\cot 2r + 2(k-1)(\cot r - \tan r)\}2\cot 2r - (2\cot 2r)^2 \\ &= 2(k-1)(2\cot 2r)^2 = 8(k-1)\cot^2 2r. \end{aligned}$$

From this, together with $m = 2k$, we obtain

$$a + b = 4(k-1)\left\{1 + 2\cot^2 2r\right\}. \tag{6.23}$$

For any X orthogonal to the vector fields ξ, $A\xi$, AN such that $SX = \cot r X$ Equation (6.22) becomes

$$aX = (4k-1)X + \{h\cot r - \cot^2 r\}X,$$

where

$$h\cot r - \cot^2 r = \{2\cot 2r + 2(k-1)2\cot 2r\}\cot r - \cot^2 r$$

$$= (2k-1)(\cot r - \tan r)\cot r - \cot^2 r$$
$$= 2(k-1)\cot^2 r - (2k-1).$$

From this, together with (6.23), we have

$$a = 2k + 2(k-1)\cot^2 r,$$
$$b = -2k + 2(k-1)\tan^2 r.$$

Putting $X = A\xi$ into (6.22), and using the properties $g(A\xi, \xi) = 0$, $A^2\xi = \xi$ and $SA\xi = 0$, we have

$$aA\xi = (2m-1)A\xi + A\xi = 2mA\xi = 4kA\xi.$$

From this, together with (6.23), it follows that $a = 4k$ and $b = -4 + 8(k-1)\cot^2 2r$. Comparing with the previous values of a and b, we conclude that

$$\cot^2 r = \frac{k}{k-1}.$$

Summing up our discussions, we conclude that the tube of radius $r = \cot^{-1}\sqrt{\frac{k}{k-1}}$ around $\mathbb{C}P^k$ in Q^{2k} is a pseudo-Einstein Hopf hypersurface in the complex quadric Q^{2k} with $\mathfrak{A}$-isotropic unit normal. Of course, the constants a and b are respectively given by $a = 4k$ and $b = -4 + \frac{2}{k}$. Because it has been calculated as follows:

$$b = -2k + 2(k-1)\tan^2 r$$
$$= -2k + \frac{2(k-1)^2}{k}$$
$$= -4 + \frac{2}{k}.$$

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